

THE INFINITELY REDUCIBLE GAUGE SYMMETRY IN UNCONSTRAINED BRST YANG-MILLS THEORY

B. SPENCE

Department of Physics, Queen Mary & Westfield College, Mile End Road London E1 4NS, UK

Received 19 June 1990

A study is made of the infinitely reducible prepotential gauge symmetry recently found in the unconstrained BRST approach to Yang-Mills theory. This symmetry has an associated set of "BRST for BRST" transformations. The unconstrained BRST approach is applied to these, and constraints upon the field strengths of suitable gauge fields are shown to yield the BRST for BRST rules. The unconstrained BRST approach in fact applies *ad infinitum*, and we give the general form of the gauge-invariant Yang-Mills action in terms of appropriate prepotentials.

1. Introduction

The BRST symmetry¹ of a quantum gauge theory is the quantum analog of the gauge symmetry of the classical theory, and is increasingly recognized as fundamental to the understanding of gauge-invariant systems.²⁻⁵ However, BRST symmetry is not a manifest symmetry of the quantum action. Given the fundamental rôle of BRST symmetry, one would clearly like a formulation of the quantum theory in which this symmetry is manifest.

Using an extra anticommuting coordinate, one can write the gauge-fixing actions for gauge theories so that the BRST invariance is explicit.^{3,6,7} However, despite a number of attempts,⁷ no satisfactory explicitly BRST-invariant formulation of the gauge-invariant actions for gauge theories had been found until recently. In the approach using the extra anticommuting coordinate, the BRST transformations can be expressed as the vanishing of certain components of the gauge field curvatures in the extended superspace. In recent work,^{8,9} these curvature constraints were solved in terms of unconstrained superspace prepotentials. If one writes the quantum action for the theory in terms of these prepotentials, then the BRST invariance of the whole action, including the original gauge-invariant action, becomes manifest. (This is described in Ref. 9 as the "unconstrained BRST" approach to quantum gauge theories. It is analogous to the approach used in supersymmetric gauge theories, where curvature constraints are solved in terms of superspace prepotentials.)

Thus, we now have explicitly BRST-invariant formulations of the quantum actions for Yang-Mills^{8,9} and other gauge theories.⁹ However, in Ref. 8 it was noted that in the unconstrained BRST description of Yang-Mills theory there is an infinitely reducible gauge symmetry, arising from prepotential gauge invariances present in the solutions to the curvature constraints. This gauge symmetry is present even after fixing of the Yang-Mills gauge invariance. In this paper we study this infinitely reducible

$$sA_\mu = \partial_\mu c + [A_\mu, c], \quad (2.5)$$

$$sc = -\frac{1}{2}[c, c].$$

In the superspace approach to BRST, the nilpotence of the BRST generator follows automatically from (2.3). (The situation where one also has anti-BRST invariance can be described using a formalism with two real anticommuting coordinates. In the unconstrained approach, this is discussed in Refs. 8 and 9. Specialization of the results in this paper to the case where anti-BRST invariance is also present is straightforward and will not be presented.)

The constraints (2.2) were solved in Refs. 8 and 9, using superspace prepotentials. In the notation of Ref. 8, the prepotentials consist of a commuting superfield $g(x, \theta)$ and an anticommuting superfield $Y_\mu(x, \theta)$, and the solutions to (2.2) are

$$\begin{aligned} \mathcal{A}_\mu &= g^{-1} \partial_\mu g + g^{-1} (\partial_\theta Y_\mu) g, \\ \mathcal{A}_\theta &= g^{-1} \partial_\theta g. \end{aligned} \quad (2.6)$$

To simplify the appearance of equations, we will henceforth adopt the notation

$$\partial := \partial_\theta. \quad (2.7)$$

The constraint solutions (2.6) are invariant under the prepotential transformations

$$\delta g = -(\partial \lambda^{(1)}) g, \quad \delta Y_\mu = \nabla_\mu \lambda^{(1)} + \partial \lambda_\mu^{(1)}, \quad (2.8)$$

where $\lambda^{(1)}(x, \theta)$ is an anticommuting superfield and $\lambda_\mu^{(1)}(x, \theta)$ a commuting one. ∇_μ , used in (2.8), is defined by

$$\nabla_\mu = \partial_\mu + [\partial Y_\mu, \cdot]. \quad (2.9)$$

The gauge symmetry (2.8) is in fact infinitely reducible, with the following infinite sequence of higher-order gauge invariances:

$$\delta \lambda^{(i)} = -\partial \lambda^{(i+1)}, \quad \delta \lambda_\mu^{(i)} = \nabla_\mu \lambda^{(i+1)} + \partial \lambda_\mu^{(i+1)} \quad (i = 1, 2, \dots), \quad (2.10)$$

where $\lambda^{(i)}, \lambda_\mu^{(i)}$ ($i = 1, 2, \dots$) are superfields whose statistics can be deduced from (2.8) and (2.10). Using this infinitely reducible gauge symmetry, one can show that the unconstrained BRST approach correctly describes the degrees of freedom of the quantum Yang-Mills theory.⁸

In order to write the Yang-Mills action in terms of the prepotentials, the following field was introduced in Ref. 8:

gauge symmetry. The question of gauge-fixing is initially considered, and the quantum action is given. This action is invariant under the BRST transformations associated with the infinitely reducible gauge symmetry. To make this symmetry explicit requires further application of the unconstrained BRST approach. Thus we are led to derive the BRST rules, associated with the infinitely reducible gauge invariances, by applying constraints to the field strengths of appropriate fields. These constraints may then be solved, using "pre-potentials." However, these solutions turn out to possess gauge invariances. We are then led to apply the unconstrained BRST approach to the BRST formulation of these symmetries, etc. In fact this procedure does not terminate, and we will give the form of the gauge-invariant action at any given stage. The unconstrained BRST approach thus reveals a surprising structure in quantum Yang-Mills theory.

2. Unconstrained BRST Yang-Mills Theory

Let us begin by recalling some results from Ref. 8. We consider a superspace with coordinates $Z^M = (x^\mu, \theta)$, where x^μ ($\mu = 1, \dots, d$) are the coordinates of d -dimensional space-time, and θ a real anticommuting coordinate. We introduce a Lie-algebra-valued superspace gauge potential, $\mathcal{A}_M(x, \theta) = (\mathcal{A}_\mu(x, \theta), \mathcal{A}_\theta(x, \theta))$. The corresponding field strength is defined by

$$\mathcal{F}_{MN} = \partial_M \mathcal{A}_N - (-1)^{MN} \partial_N \mathcal{A}_M + [\mathcal{A}_M, \mathcal{A}_N]. \quad (2.1)$$

We define $(-1)^{MN}$ to be 1 unless both M and N are indices which refer to anticommuting coordinates, in which case it is -1 . The generators t_a of the Lie algebra satisfy $[t_a, t_b] = f_{ab}^c t_c$, f_{ab}^c being the structure constants of the algebra. For any two Lie-algebra-valued fields $U = U^a t_a$, $V = V^a t_a$, we have $[U, V] = f_{ab}^c U^a V^b t_c$.

To obtain the BRST rules, we first impose the following constraints upon the field strengths:

$$\mathcal{F}_{\theta\theta} = 0, \quad \mathcal{F}_{\mu\theta} = 0. \quad (2.2)$$

If we identify the BRST generator s as

$$s = \partial_\theta = \frac{\partial}{\partial \theta} \quad (2.3)$$

and identify the Yang-Mills space-time gauge field $A_\mu(x)$ and ghost field $c(x)$ by

$$\begin{aligned} A_\mu(x) &= \mathcal{A}_\mu(x, 0), \\ c(x) &= \mathcal{A}_\theta(x, 0), \end{aligned} \quad (2.4)$$

then the constraints (2.2) (evaluated at $\theta = 0$) express the BRST rules for the space-time fields

$$G_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu + \frac{1}{2} [\partial Y_\mu, Y_\nu] - \frac{1}{2} [\partial Y_\nu, Y_\mu]. \quad (2.11)$$

This satisfies

$$\partial G_{\mu\nu} = g \mathcal{F}_{\mu\nu} g^{-1}. \quad (2.12)$$

The Yang-Mills action may then be written as

$$S^{\text{inv}} = -\frac{1}{4} \text{Tr} \int d^4x \partial G_{\mu\nu} \partial G^{\mu\nu}, \quad (2.13)$$

which is manifestly BRST-invariant, since it is written in superspace, and the BRST generator is simply given by the θ translation operator, (2.3). [(2.13) is the usual Yang-Mills action, because by (2.11), (2.1) and the first equation of (2.4), $\text{Tr}(\partial G_{\mu\nu} \partial G^{\mu\nu})|_{\theta=0} = \text{Tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu})|_{\theta=0} = \text{Tr}(F_{\mu\nu} F^{\mu\nu})$, where $F_{\mu\nu}$ is the usual space-time Yang-Mills field strength. S^{inv} is invariant under the prepotential gauge transformations (2.8), since it may be written in terms of $\mathcal{F}_{\mu\nu}$, which is invariant under (2.8) because \mathcal{A}_μ is.]

The Yang-Mills gauge invariance is expressed in BRST superspace by⁸

$$\delta g = g \Gamma, \quad \delta Y_\mu = 0, \quad (2.14)$$

where $\Gamma(x, \theta)$ is a commuting superfield. In the quantum Yang-Mills action, the gauge-fixing terms for the Yang-Mills gauge invariance may be described by a superspace gauge fermion, $\Psi(x, \theta)$. The quantum Yang-Mills action is given by

$$S_Q = S^{\text{inv}} + \int d^4x d\theta \Psi. \quad (2.15)$$

We require that S_Q be invariant under the prepotential gauge transformations (2.8), as these symmetries are a necessary part of the unconstrained BRST description of quantum Yang-Mills theory, as mentioned above. A general class of superspace gauge-fixing conditions is given by $f_\alpha(\mathcal{A}_\mu(x, \theta)) = 0$, for some set of functionals f_α (labelled by some index α). Note that these preserve the prepotential gauge invariances (2.8), as they are functionals of \mathcal{A}_μ , which is invariant under these transformations. We introduce antighost superfields $\bar{C}^\alpha(x, \theta)$, with the space-time antighost fields $\bar{C}^\alpha(x)$ and Nakanishi-Laurup (NL) fields $\pi^\alpha(x)$, given by $\bar{C}^\alpha(x) = \bar{C}^\alpha(x, 0)$, $\pi^\alpha(x) = \partial \bar{C}^\alpha(x, \theta)$. We may then take the superspace gauge fermion to be

$$\Psi = \bar{C}^\alpha f_\alpha(\mathcal{A}_\mu) + \beta \partial \bar{C}_\alpha, \quad (2.16)$$

where β is some constant. Performing the θ integrals in (2.15) gives us back the usual

quantum Yang-Mills action arising from the gauge-fixing conditions $f_\alpha(\mathcal{A}_\mu(x)) = 0$ (see Ref. 8).

The action (2.15) then has the prepotential gauge invariances (2.8), and these have the higher-order invariances (2.10). (The superfields \bar{C}^α are taken to be invariant under the prepotential gauge transformations.) These gauge invariances need to be fixed in order to define a correct quantum action. In general, the resulting gauge-fixed quantum action is given by (cf. the above)

$$S_Q^{(2)} = S_Q + s_2 \Psi^{(2)}, \quad (2.17)$$

where S_Q is given in (2.15), $\Psi^{(2)}$ is the gauge fermion which fixes the prepotential gauge invariances, and s_2 is the BRST type generator which is associated with the prepotential gauge transformations (this will be studied in the following). Inspection of (2.8) and (2.10) suggests the gauge-fixing conditions

$$\begin{aligned} \partial \partial^\mu Y_\mu &= \theta \lambda^{(1)} = 0, \\ \theta Y_\mu &= \theta \lambda_\mu^{(1)} = 0. \end{aligned} \quad (2.18)$$

The gauge fermion $\Psi^{(2)}$ corresponding to this choice is (see Ref. 8)

$$\Psi^{(2)} = \int d^4x d\theta \left[\bar{C}_\mu^{(1)} \theta Y^\mu - \bar{C}^{(1)} \partial \partial^\mu Y_\mu + \sum_{i=1}^{\infty} (\bar{C}^{(i+1)} \theta C^{(i)\mu} + \bar{C}^{(i+1)} \theta C^{(i)}) \right]. \quad (2.19)$$

However, this choice breaks the BRST symmetry—as one can see from the presence of explicit θ 's in (2.19) (recalling that the BRST transformations are generated by translations in θ). Nevertheless, the quantum action (2.17), arising from this or another gauge-fixing prescription, has a BRST invariance corresponding to the gauge transformation rules (2.8) and (2.10). We will call this "BRST for BRST." The usual quantum Yang-Mills action is BRST-invariant, but not explicitly so—the point of the work summarized earlier was to make this invariance explicit by rewriting the gauge-invariant action in terms of the superspace prepotentials g and Y_μ , which solved the BRST curvature constraints (2.2). This situation arises again with regard to "BRST for BRST"—the action (2.17) will be invariant under the BRST for BRST rules corresponding to the gauge transformations (2.8) and (2.10), but this will not be explicit. In line with the philosophy mentioned in the Introduction, we wish to make this BRST for BRST invariance explicit. To do this, we will first express the BRST for BRST rules—corresponding to (2.8) and (2.10)—as constraints upon the field strengths of gauge fields in an extended superspace. These constraints will then be solved in terms of "pre-potentials" and the quantum BRST for BRST action written in terms of the pre-potentials. This quantum action will be explicitly invariant under the BRST for BRST transformations. We will show how this works in the following. Of course, the question immediately arises as to whether this quantum BRST for BRST-invariant action will have pre-potential gauge invariances, analo-

gous to the prepotential gauge invariances (2.8) and (2.10) of the quantum BRST-invariant Yang-Mills action (2.15). This proves to be the case, and one can thus investigate a (BRST)³ formalism, etc. It turns out that this procedure does not terminate, leading to a (BRST)ⁿ formulation of the quantum Yang-Mills action, for any positive integer n .

3. Abelian BRST for BRST

The main features of the BRST for BRST approach are exhibited in the Abelian case, which we will discuss first. The solutions to the Abelian versions of the constraints (2.2) are

$$\begin{aligned} \mathcal{A}_\mu &= \partial_\mu \alpha + \partial Y_\mu, \\ \mathcal{A}_\theta &= \partial \alpha, \end{aligned} \quad (3.1)$$

with α a commuting superfield and Y_μ an anticommuting one. The solutions (3.1) have the prepotential gauge invariances

$$\begin{aligned} \delta \alpha &= -\partial \lambda^{(1)}, \\ \delta Y_\mu &= \partial_\mu \lambda^{(1)} + \partial \lambda_\mu^{(1)}, \end{aligned} \quad (3.2)$$

and we have the higher-order gauge invariances ($i = 1, 2, \dots$)

$$\begin{aligned} \delta \lambda^{(i)} &= -\partial \lambda^{(i+1)}, \\ \delta \lambda_\mu^{(i)} &= \partial_\mu \lambda^{(i+1)} + \partial \lambda_\mu^{(i+1)}, \end{aligned} \quad (3.3)$$

(3.2) and (3.3) being the Abelian versions of (2.8) and (2.10).

We introduce a set of commuting ghosts $c^{(i)}$, and anticommuting ghosts $c_\mu^{(i)}$ ($i = 1, 2, \dots$) corresponding to the gauge parameters. The following transformations are nilpotent, $(s_2)^2 = 0$, and may be taken to be the BRST for BRST rules corresponding to the gauge invariances (3.2) and (3.3):

$$\begin{aligned} s_2 \alpha &= \partial c^{(1)}, \\ s_2 Y_\mu &= \partial_\mu c^{(1)} - \partial c_\mu^{(1)}, \\ s_2 c^{(i)} &= (-1)^i \partial c^{(i+1)}, \\ s_2 c_\mu^{(i)} &= \partial_\mu c^{(i+1)} - (-1)^i \partial c_\mu^{(i+1)}. \end{aligned} \quad (3.4)$$

(We use the notation s_2 for the generator of these BRST for BRST transformations. This is to avoid confusion with the BRST generator s discussed in the previous

section.) We will now deduce the rules (3.4) by imposing constraints on field strengths, using a similar procedure to that of (2.2)–(2.5). To do this we will introduce an extra anticommuting variable θ_2 , in addition to θ and x^μ . The BRST rules (3.4) will be obtained with the identification $s_2 = \partial/\partial\theta_2$. We will begin by defining field strengths for the fields α and Y_μ . These field strengths must be invariant (as we are discussing the Abelian case) under the gauge transformations. In order to define field strengths for α and Y_μ , it proves necessary to introduce a set of auxiliary fields, as we will now explain.

The field strength for α is taken to be \mathcal{A}_μ [defined in (3.1)]. This is invariant under the gauge transformations (3.2), as required. We now seek a field strength for the field Y_μ . The expression $2\partial_{[\mu} Y_{\nu]}$ is invariant under the $\lambda^{(1)}$ symmetry of (3.2), but not under the $\lambda_\mu^{(1)}$ symmetry—we cannot therefore take this as the Abelian field strength for Y_μ . However, if we introduce a new field, $X_{\mu\nu}$, a rank-2 antisymmetric tensor, which transforms as

$$\delta X_{\mu\nu} = 2\partial_{[\mu} \lambda_{\nu]}^{(1)} \quad (3.5)$$

(antisymmetrization of indices carries unit weight), then the field $H_{\mu\nu}$, defined by

$$H_{\mu\nu} = 2\partial_{[\mu} Y_{\nu]} - \partial X_{\mu\nu}, \quad (3.6)$$

is invariant under the $\lambda^{(1)}$ and $\lambda_\mu^{(1)}$ symmetries and may be taken to be the field strength for the Y_μ gauge field. However, (3.5) is not invariant if we transform $\lambda_\mu^{(1)}$ by

$$\delta \lambda_\mu^{(1)} = \partial_\mu \lambda^{(2)} + \partial \lambda_\mu^{(2)}, \quad (3.7)$$

which is amongst the transformations (3.3). We wish to preserve all gauge invariances, so we will also introduce a gauge parameter rank-2 antisymmetric tensor field, $\lambda_{\mu\nu}^{(1)}$, transforming as

$$\delta \lambda_{\mu\nu}^{(1)} = 2\partial_{[\mu} \lambda_{\nu]}^{(2)} \quad (3.8)$$

Then, instead of (3.5) we will take

$$\delta X_{\mu\nu} = 2\partial_{[\mu} \lambda_{\nu]}^{(1)} - \partial \lambda_{\mu\nu}^{(1)}, \quad (3.9)$$

which is invariant under (3.7) and (3.8). However, (3.8) is not invariant under $\delta \lambda_\mu^{(2)} = \partial_\mu \lambda^{(3)} + \partial \lambda_\mu^{(3)}$, another of the symmetries (3.3). Thus we need to introduce a further antisymmetric tensor field, $\lambda_{\mu\nu}^{(2)}$, with $\delta \lambda_{\mu\nu}^{(2)} = 2\partial_{[\mu} \lambda_{\nu]}^{(3)}$, and replace (3.8) by $\delta \lambda_{\mu\nu}^{(1)} = 2\partial_{[\mu} \lambda_{\nu]}^{(2)} - \partial \lambda_{\mu\nu}^{(2)}$, which is then invariant. But then the variation of $\lambda_{\mu\nu}^{(2)}$ is not invariant under the $\lambda^{(4)}$ symmetries, etc. The end result is that one introduces a set of new antisymmetric tensor gauge parameter fields, $\lambda_{\mu\nu}^{(i)}$ ($i = 1, 2, \dots$), transforming as

$$\delta\lambda_{\mu\nu}^{(i)} = 2\partial_{[\mu}\lambda_{\nu]}^{(i+1)} - \partial\lambda_{\mu\nu}^{(i+1)}. \quad (3.10)$$

Now, in (3.10), if we vary $\lambda_{\mu}^{(i+1)}$, using the second equation of (3.3) with $i \rightarrow (i+1)$, and vary $\lambda_{\mu\nu}^{(i+1)}$ using (3.10) with $i \rightarrow (i+1)$, then the result is zero. In this sense the gauge transformations (3.10) preserve the invariances (3.3).

We note in passing that, from the Abelian version (2.13) and (3.6),

$$-\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} = - \int d^4x d\theta \partial_{[\mu} Y_{\nu]} \partial \partial^{\mu} Y^{\nu} = -\frac{1}{4} \int d^4x d\theta H_{\mu\nu} \partial H^{\mu\nu}, \quad (3.11)$$

so that the gauge-invariant Abelian action can be written in terms of the prepotential field strength $H_{\mu\nu}$.

Now we seek a field strength for the new gauge field $X_{\mu\nu}$. The obvious guess for this, i.e. $\partial_{[\mu} X_{\nu\rho]}$, is not invariant under (3.9). Thus, we will introduce another field, $X_{\mu\nu\rho}$, a rank-3 antisymmetric tensor, which transforms as

$$\delta X_{\mu\nu\rho} = 3\partial_{[\mu}\lambda_{\nu\rho]}^{(i)}. \quad (3.12)$$

Now we can define the following field strength for $X_{\mu\nu}$:

$$H_{\mu\nu\rho} = 3\partial_{[\mu} X_{\nu\rho]} + \partial X_{\mu\nu\rho}, \quad (3.13)$$

which is invariant under (3.9) and (3.12).

However, the variation (3.12) is not invariant under the transformations (3.10) (with $i=1$). This can be remedied with the introduction of a set of rank-3 antisymmetric tensor gauge parameter fields, $\lambda_{\mu\nu\rho}^{(i)}$ ($i=1, 2, \dots$), whose gauge transformation rules will be given below [this is analogous to the situation with (3.5), and the introduction of the gauge parameter superfields $\lambda_{\mu\nu}^{(i)}$]. We next seek a field strength for the field $X_{\mu\nu\rho}$. This will involve the introduction of a new rank-4 antisymmetric tensor field, $X_{\mu\nu\rho\sigma}$, as well as new antisymmetric tensor gauge parameter fields, $\lambda_{\mu\nu\rho\sigma}^{(i)}$ ($i=1, 2, \dots$). The field strength for $X_{\mu\nu\rho\sigma}$ will be defined using a new antisymmetric tensor field, $X_{\mu\nu\rho\sigma}$, etc. The following structure emerges:

There are antisymmetric tensor gauge fields $X := \alpha$, $X_{\mu,1} := Y_{\mu,1}$, $X_{\mu,1,2}$, $X_{\mu,1,2,3}$, \dots , $X_{\mu,1,2,3,4}$, \dots , and antisymmetric tensor gauge parameter fields $\lambda^{(i)}$, $\lambda_{\mu}^{(i)}$, $\lambda_{\mu,1}^{(i)}$, $\lambda_{\mu,1,2}^{(i)}$, \dots , $\lambda_{\mu,1,2,3}^{(i)}$, \dots ($n=0, 1, \dots; i=1, 2, \dots$). The gauge transformation rules are

$$\delta X_{\mu,1,\dots,\mu_n} = n\partial_{[\mu,1}\lambda_{\mu_2,\dots,\mu_n]}^{(i)} - (-1)^n \partial\lambda_{\mu,1,\dots,\mu_n}^{(i)}, \quad (3.14)$$

$$\delta\lambda_{\mu,1,\dots,\mu_n}^{(i)} = n\partial_{[\mu,1}\lambda_{\mu_2,\dots,\mu_n]}^{(i+1)} - (-1)^n \partial\lambda_{\mu,1,\dots,\mu_n}^{(i+1)}. \quad (3.15)$$

The field strengths for the gauge fields $X_{\mu,1,\dots,\mu_n}$ are given by

$$H_{\mu,1,\dots,\mu_{n+1}} = (n+1)\partial_{[\mu,1}\lambda_{\mu_2,\dots,\mu_{n+1}]}^{(i+1)} - (-1)^{n+1}\partial X_{\mu,1,\dots,\mu_{n+1}}. \quad (3.16)$$

These field strengths are invariant under the transformations given in (3.14). Similarly, the variations (3.14) are invariant under (3.15) with $i=1$, and the variations (3.15) are invariant under the transformations (3.15) with $i \rightarrow (i+1)$.

Having defined invariant field strengths for the prepotentials α and Y_{μ} , and the auxiliary gauge fields $X_{\mu\nu}$, etc., we are now in a position to deduce the BRST for BRST rules, associated with the transformations (3.14) [these will include the transformations postulated in (3.4)], from field strength constraints in an extended superspace. [This approach is the BRST for BRST analog of the procedure used in (2.1)-(2.5) for the BRST transformations. In Sec. 5 we will pursue the BRST for BRST analog of the unconstrained BRST approach detailed in the other part of Sec. 2.]

We first introduce the additional anticommuting variable θ_2 , supplementing x and θ . The BRST for BRST generator s_2 will be identified with translations in θ_2 , i.e.

$$s_2 = \frac{\partial}{\partial\theta_2}. \quad (3.17)$$

We want to generalize the tensor fields $X_{\mu,1,\dots,\mu_n}$, used in the above, to the superspace with coordinates (x, θ, θ_2) . This is most simply done by using the language of differential forms. For a general antisymmetric tensor $X_{\mu,1,\dots,\mu_n}(x, \theta)$, we will define the form

$$X_n = \frac{1}{n!} dx^{\mu_1} \dots dx^{\mu_n} X_{\mu_1,\dots,\mu_n}, \quad (3.18)$$

where the subscript n on X indicates the degree of the form (this subscript may be dropped in circumstances where this does not cause confusion). The differential operator d is defined by

$$d = dx^\mu \partial_\mu. \quad (3.19)$$

Given a form $X_n(x, \theta)$, we will define a new form $\tilde{X}_n(x, \theta, \theta_2)$ as follows:

$$\tilde{X}_n(x, \theta, \theta_2) = \frac{1}{n!} dZ^{\mu_1} \dots dZ^{\mu_n} \tilde{X}_{\mu_1,\dots,\mu_n}(x, \theta, \theta_2), \quad (3.20)$$

where the upper case Roman indices run over (μ, θ_2) , with $Z^M = (x^\mu, \theta_2)$. We will also define

$$\tilde{d} = d + d\theta_2 \partial_{\theta_2}. \quad (3.21)$$

[Notice that there are no $d\theta$ terms in (3.20) and (3.21). Thus, we are treating θ as an

"internal" variable, and our forms are really differential forms in the (x, θ_2) superspace.]

Now let us see how these definitions, together with (3.17), enable us to derive the BRST for BRST rules corresponding to the gauge transformations (3.14) and (3.15). The procedure is to generalize the field strength tensors (3.16) to tilded tensors \tilde{H} . Then we will impose constraints upon the components of \tilde{H} . The tensors \tilde{H} will be defined in the same way as the tensors H , except that tilded quantities will replace those without tildes, using (3.20) and (3.21). This procedure ensures that the field strengths \tilde{H} obey superspace analogs of the Bianchi identities which the field strengths H obey.

The simplest case occurs with the field strength for the gauge field α , which is $\mathcal{A}_\mu = \partial_\mu \alpha + \partial Y_\mu$. The x space one-form associated with this is $\mathcal{A}_1(x, \theta) = d\alpha + \partial Y_1$, where $Y_1 = dx^\mu Y_\mu$. Using (3.18)–(3.21), we define

$$\tilde{\mathcal{A}}_1(x, \theta, \theta_2) = dx^\mu \tilde{\mathcal{A}}_\mu + d\theta_2 \tilde{\mathcal{A}}_{\theta_2} = \tilde{d}\alpha + \partial \tilde{Y}_1, \quad (3.22)$$

where

$$\tilde{Y}_1 = dx^\mu \tilde{Y}_\mu + d\theta_2 \tilde{Y}_{\theta_2}, \quad (3.23)$$

with $\tilde{Y}_{\theta_2}(x, \theta, \theta_2)$ a new (commuting) superfield. [Note that all component fields of tilded forms are functions of (x, θ, θ_2) , as we see in general in (3.20).] From (3.22) we learn that

$$\tilde{\mathcal{A}}_{\theta_2} = \partial_{\theta_2} \tilde{\alpha} - \partial \tilde{Y}_{\theta_2}. \quad (3.24)$$

Now, we identify the (x, θ) superfields α , Y_μ , \mathcal{A}_μ , and the ghost $c^{(1)}$ by

$$\alpha(x, \theta) = \tilde{\alpha}(x, \theta, 0), \quad Y_\mu(x, \theta) = \tilde{Y}_\mu(x, \theta, 0), \quad (3.25)$$

$$\mathcal{A}_\mu(x, \theta) = \tilde{\mathcal{A}}_\mu(x, \theta, 0), \quad c^{(1)}(x, \theta) = \tilde{Y}_{\theta_2}(x, \theta, 0).$$

By the use of (3.25), the field strength constraint

$$\tilde{\mathcal{A}}_{\theta_2}(x, \theta, \theta_2) = 0 \quad (3.26)$$

evaluated at $\theta_2 = 0$, together with (3.17), gives the BRST for BRST transformation

$$s_2 \alpha = \partial c^{(1)}, \quad (3.27)$$

which is the first equation of (3.4). [The coefficient of θ_2 in the superspace expansion of the constraint (3.26) gives the BRST for BRST transform of (3.27). This follows from (3.7).]

The next gauge field is Y_μ , whose field strength $H_{\mu\nu}$ is given in (3.6). Writing this

equation in differential form notation, we have

$$H_2 = dY_1 - \partial X_2. \quad (3.28)$$

The field \tilde{H}_2 is then defined by

$$\tilde{H}_2 = \tilde{d}Y_1 - \partial \tilde{X}_2, \quad (3.29)$$

with \tilde{Y}_1 given in (3.23) and

$$\tilde{H}_2 = \frac{1}{2} dx^\mu dx^\nu \tilde{H}_{\mu\nu} + d\theta_2 dx^\mu \tilde{H}_{\theta_2\mu} + \frac{1}{2} d\theta_2 d\theta_2 \tilde{H}_{\theta_2\theta_2}, \quad (3.30)$$

and a similar expansion for \tilde{X}_2 . The components are then given by $(\tilde{H}_{\theta_2\mu} = -\tilde{H}_{\mu\theta_2})$, and

$$\begin{aligned} \tilde{H}_{\mu\nu} &= \partial_\mu \tilde{Y}_\nu - \partial_\nu \tilde{Y}_\mu - \partial \tilde{X}_{\mu\nu}, \\ \tilde{H}_{\theta_2\mu} &= \partial_{\theta_2} \tilde{Y}_\mu - \partial_\mu \tilde{Y}_{\theta_2} + \partial \tilde{X}_{\theta_2\mu}, \end{aligned} \quad (3.31)$$

$$\tilde{H}_{\theta_2\theta_2} = -2\partial_{\theta_2} \tilde{Y}_{\theta_2} - \partial \tilde{X}_{\theta_2\theta_2}.$$

Now we make the identifications

$$\begin{aligned} X_{\mu\nu}(x, \theta) &= \tilde{X}_{\mu\nu}(x, \theta, 0), \\ c_\mu^{(1)}(x, \theta) &= \tilde{X}_{\theta_2\mu}(x, \theta, 0), \\ c^{(2)}(x, \theta) &= \frac{1}{2} \tilde{X}_{\theta_2\theta_2}(x, \theta, 0). \end{aligned} \quad (3.32)$$

[The identifications (3.25) and (3.32), and others to follow, are the BRST for BRST analogs of the BRST identifications (2.4).] The constraints

$$\tilde{H}_{\theta_2\mu}(x, \theta, \theta_2) = \tilde{H}_{\theta_2\theta_2}(x, \theta, \theta_2) = 0 \quad (3.33)$$

evaluated at $\theta_2 = 0$, together with (3.17), then imply the BRST for BRST transformations

$$\begin{aligned} s_2 Y_\mu &= \partial_\mu c^{(1)} - \partial c_\mu^{(1)}, \\ s_2 c^{(1)} &= -\partial c^{(2)}, \end{aligned} \quad (3.34)$$

which are amongst the transformations (3.4).

We now turn to the gauge field $X_{\mu\nu}$. Its field strength is given in (3.13), which can be written in differential form notation as

$$H_3 = dX_2 + \partial X_3. \quad (3.35)$$

Thus we define

$$\tilde{H}_3 = d\tilde{X}_2 + \partial\tilde{X}_3. \quad (3.36)$$

In components this gives

$$\begin{aligned} \tilde{H}_{\mu\nu\rho} &= 3\partial_{[\mu}\tilde{X}_{\nu\rho]} + \partial\tilde{X}_{\mu\nu\rho}, \\ \tilde{H}_{\theta_1\mu\nu} &= 2\partial_{[\mu}\tilde{X}_{\nu]\theta_1} + \partial_{\theta_1}\tilde{X}_{\mu\nu} - \partial\tilde{X}_{\theta_1\mu\nu}, \\ \tilde{H}_{\theta_1\theta_2\mu} &= \partial_{\mu}\tilde{X}_{\theta_1\theta_2} - 2\partial_{\theta_1}\tilde{X}_{\theta_2\mu} + \partial\tilde{X}_{\theta_1\theta_2\mu}, \end{aligned} \quad (3.37)$$

$$\tilde{H}_{\theta_1\theta_2\theta_3} = 3\partial_{\theta_1}\tilde{X}_{\theta_2\theta_3} - \partial\tilde{X}_{\theta_1\theta_2\theta_3}.$$

Defining

$$\begin{aligned} X_{\mu\nu\rho}(x, \theta) &= \tilde{X}_{\mu\nu\rho}(x, \theta, 0), & c_{\mu\nu}^{(1)}(x, \theta) &= \tilde{X}_{\theta_1\mu\nu}(x, \theta, 0), \\ c_{\mu}^{(2)}(x, \theta) &= \frac{1}{2}\tilde{X}_{\theta_1\theta_2\mu}(x, \theta, 0), & c^{(3)}(x, \theta) &= \frac{1}{6}\tilde{X}_{\theta_1\theta_2\theta_3}(x, \theta, 0), \end{aligned} \quad (3.38)$$

we find that the constraints

$$\tilde{H}_{\theta_1\mu\nu} = \tilde{H}_{\theta_1\theta_2\mu} = \tilde{H}_{\theta_1\theta_2\theta_3} = 0 \quad (3.39)$$

evaluated at $\theta_2 = 0$ give the BRST for BRST transformations

$$\begin{aligned} s_2 X_{\mu\nu} &= 2\partial_{[\mu} c_{\nu]}^{(1)} + \partial c_{\mu\nu}^{(1)}, \\ s_2 c_{\mu}^{(1)} &= \partial_{\mu} c^{(2)} + \partial c_{\mu}^{(2)}, \\ s_2 c^{(2)} &= \partial c^{(3)}. \end{aligned} \quad (3.40)$$

[The first equation of (3.40) is the BRST for BRST transformation associated with (3.9), the ghost $c_{\mu\nu}^{(1)}$ being associated with the gauge transformation parameter field $\lambda_{\mu\nu}^{(1)}$. The other two equations are amongst those of (3.4).]

Having seen how this procedure works in detail in the simpler cases, let us now give the general derivation of the BRST for BRST transformations corresponding to the gauge transformations (3.14) and (3.15), using the field strengths (3.16). First, we

write these equations in differential form notation as

$$\delta X_n = d\lambda_{n-1}^{(1)} - (-1)^n \partial \lambda_n^{(1)}, \quad (3.41)$$

$$\delta \lambda_n^{(1)} = d\lambda_{n-1}^{(1)} - (-1)^n \partial \lambda_n^{(1)}, \quad (3.42)$$

and

$$H_{n+1} = dX_n - (-1)^{n+1} \partial X_{n+1}. \quad (3.43)$$

We define $\tilde{H}_{n+1}(x, \theta, \theta_2)$ by

$$\tilde{H}_{n+1} = d\tilde{X}_n - (-1)^{n+1} \partial\tilde{X}_{n+1}. \quad (3.44)$$

Each gauge parameter field $\lambda_n^{(1)}(x, \theta)$ has an associated ghost field $c_n^{(1)}(x, \theta)$, which is anticommuting if $(-1)^n$ is odd, and commuting if it is even. These ghosts are defined by

$$c_{\mu_1 \dots \mu_n}^{(1)}(x, \theta) = \frac{1}{n!i!} \tilde{X}_{\theta_2 \dots \theta_{n+1} \mu_1 \dots \mu_n}(x, \theta, 0), \quad (3.45)$$

where there are i θ_2 subscripts on the field \tilde{X} on the right-hand side. The fields $\tilde{X}_{\theta_2 \dots \theta_{n+1} \mu_1 \dots \mu_n}(x, \theta, 0)$ are the components of the form $\tilde{X}_{n+1}(x, \theta, \theta_2)$, evaluated at $\theta_2 = 0$. Now we impose the constraints

$$\tilde{H}_{\theta_2 \mu_1 \dots \mu_n} = \tilde{H}_{\theta_2 \theta_1 \mu_1 \dots \mu_{n-1}} = \dots = \tilde{H}_{\theta_2 \theta_1 \dots \theta_{n-1} \mu} = \tilde{H}_{\theta_2 \theta_1 \dots \theta_n} = 0. \quad (3.46)$$

With the identifications (3.45) and $X_{\mu_1 \dots \mu_n}(x, \theta) = \tilde{X}_{\mu_1 \dots \mu_n}(x, \theta, 0)$, and by the use of (3.44), the constraints (3.46) evaluated at $\theta_2 = 0$ give the BRST for BRST transformation rules

$$\begin{aligned} s_2 X_{\mu_1 \dots \mu_n} &= n\partial_{[\mu_1} c_{\mu_2 \dots \mu_n]}^{(1)} + (-1)^n \partial c_{\mu_1 \dots \mu_n}^{(1)}, \\ s_2 c_{\mu_1 \dots \mu_n}^{(1)} &= n\partial_{[\mu_1} c_{\mu_2 \dots \mu_n]}^{(1+1)} + (-1)^{n+1} \partial c_{\mu_1 \dots \mu_n}^{(1+1)}, \end{aligned} \quad (3.47)$$

where $i = 1, 2, \dots, n = 0, 1, \dots$. It is easy to check that $(s_2)^2 = 0$. Note that the transformations (3.47) include those proposed earlier, (3.4).

Now, let us note that it is of course true that a rank- $(d+1)$ antisymmetric tensor in d dimensions must be zero. Thus, in order for all the antisymmetric tensors introduced above to be nonzero, we need to introduce a fictitious infinity of extra (bosonic) dimensions. This is a device necessary for deriving the BRST rules from curvature constraints, and after this is done, one drops the fictitious bosonic dimensions, together with any remaining antisymmetric tensors of rank larger than d .

With this in mind, we conclude that, in d dimensions, the full set of BRST for BRST transformations is given by (3.47), with $i = 1, 2, \dots, \infty$, and $n = 0, 1, \dots, d$. The full set of gauge fields and of ghosts are $X, X_\mu, \dots, X_{\mu_1 \dots \mu_d}$ and $c^{(i)}, c_\mu^{(i)}, \dots, c_{\mu_1 \dots \mu_d}^{(i)}$. As discussed in Ref. 8, antighost and Nakanishi-Lautrup (NL) fields must also be introduced (these arise naturally in the formulation with anti-BRST as well as BRST invariance). We introduce antighost and NL superfields $\bar{c}^{(i)}_{\mu_1 \dots \mu_n}, \pi^{(i)}_{\mu_1 \dots \mu_n}$ ($i = 1, 2, \dots, \infty; n = 0, 1, \dots, d$), with BRST for BRST rules

$$\begin{aligned} s_2 \bar{c}^{(i)}_{\mu_1 \dots \mu_n} &= \pi^{(i)}_{\mu_1 \dots \mu_n}, \\ s_2 \pi^{(i)}_{\mu_1 \dots \mu_n} &= 0. \end{aligned} \quad (3.48)$$

(The superfield $\bar{c}^{(i)}_{\mu_1 \dots \mu_n}$ has the same statistics as $c^{(i)}_{\mu_1 \dots \mu_n}$, while the superfield $\pi^{(i)}_{\mu_1 \dots \mu_n}$ has the opposite statistics to $c^{(i)}_{\mu_1 \dots \mu_n}$.)

From (2.13) and (2.16), we see that only X (where $g = e^X$ here) and X_μ ($:= Y_\mu$) appear in the prepotential gauge transformation invariant quantum Abelian action. Together with the fields $c^{(i)}, c_\mu^{(i)}, \bar{c}^{(i)}, \bar{c}_\mu^{(i)}, \pi^{(i)}, \pi_\mu^{(i)}$, these form a subalgebra with respect to the transformations (3.47) and (3.48). The remaining fields we may consider as an additional set of auxiliary fields. Note, however, that these auxiliary fields are necessary in order to define field strengths for the gauge fields, and hence derive the BRST for BRST rules, as we have discussed above.

4. Non-Abelian BRST for BRST

Let us now turn to the non-Abelian generalizations of the results of the previous section. We begin by seeking field strengths for the gauge fields, generalizing (3.43) to the non-Abelian case.

First, let us write the constraint solutions (2.6), using differential form notation, as

$$\mathcal{A}_1 = g^{-1} dg + g^{-1} \partial Y g, \quad (4.1)$$

$$\mathcal{A}_g = g^{-1} \partial g, \quad (4.2)$$

where $\mathcal{A}_1 = dx^\mu \mathcal{A}_\mu$ and $Y = dx^\mu Y_\mu$. The prepotential gauge invariances (2.8) can be written as

$$\begin{aligned} (\delta g) g^{-1} &= -\partial \lambda^{(1)}, \\ \delta Y &= \nabla \lambda^{(1)} + \partial \lambda_1^{(1)}, \end{aligned} \quad (4.3)$$

where $\nabla = dx^\mu \nabla_\mu$ and $\lambda_1^{(1)} = dx^\mu \lambda_{1\mu}^{(1)}$. The higher-order invariances (2.10) are similarly written:

$$\begin{aligned} \delta \lambda^{(i)} &= -\partial \lambda^{(i+1)}, \\ \delta \lambda_1^{(i)} &= \nabla \lambda^{(i+1)} + \partial \lambda_1^{(i+1)} \quad (i = 1, 2, \dots), \end{aligned} \quad (4.4)$$

where $\lambda_1^{(i)} = dx^\mu \lambda_{1\mu}^{(i)}$.

We will take \mathcal{A}_1 , defined by (4.1), to be the field strength associated with the gauge field g . Now we will find a field strength $H_2 = \frac{1}{2} dx^\mu dx^\nu H_{\mu\nu}$ for Y . This must transform covariantly under the $\lambda^{(1)}$ and $\lambda_1^{(1)}$ gauge transformations. In the latter case this is an Abelian symmetry, so H_2 must be invariant under the $\lambda_1^{(1)}$ transformations. In the former case we may deduce the covariant gauge transformation rule by consideration of the covariant derivative ∇_μ , (2.9). Thus we require that H_2 transform under the $\lambda^{(1)}$ and $\lambda_1^{(1)}$ symmetries as

$$\delta H_2 = [\partial H_2, \lambda^{(1)}]. \quad (4.5)$$

First consider

$$G_2 = dY + \frac{1}{2} [\partial Y, Y] = \frac{1}{2} dx^\mu dx^\nu G_{\mu\nu}, \quad (4.6)$$

where $G_{\mu\nu}$ is defined by (2.11). This transforms under the $\lambda^{(1)}$, $\lambda_1^{(1)}$ gauge transformations as

$$\delta G_2 = [\partial G_2, \lambda^{(1)}] + \partial \left(\nabla \lambda_1^{(1)} - \frac{1}{2} [Y, \delta Y] \right), \quad (4.7)$$

where $\delta Y = \nabla \lambda^{(1)} + \partial \lambda_1^{(1)}$. As in the Abelian case, we thus introduce a new field, $X_2 = \frac{1}{2} dx^\mu dx^\nu X_{\mu\nu}$. Here it transforms under the $\lambda^{(1)}$, $\lambda_1^{(1)}$ gauge transformations as

$$\delta X_2 = \nabla \lambda_1^{(1)} - \frac{1}{2} [Y, \delta Y]. \quad (4.8)$$

Then the following definition of H_2 transforms covariantly [i.e. as in (4.5)]:

$$H_2 = dY + \frac{1}{2} [\partial Y, Y] - \partial X_2. \quad (4.9)$$

We note in passing that the gauge-invariant Yang-Mills action can be written in terms of the field strength H_2 —since $\partial H_{\mu\nu} = \partial G_{\mu\nu}$, whence from (2.13),

$$S^{\text{inv}} = -\frac{1}{4} \text{Tr} \int d^d x (\partial H_{\mu\nu} \partial H^{\mu\nu})|_{\theta=0} = -\frac{1}{4} \text{Tr} \int d^d x d\theta H_{\mu\nu} \partial H^{\mu\nu}. \quad (4.10)$$

Now, (4.8) does not transform covariantly under the second-order gauge transformations $\delta\lambda^{(1)} = -\partial\lambda^{(2)}$, $\delta\lambda_1^{(1)} = \nabla\lambda^{(2)} + \partial\lambda_1^{(2)}$, which are amongst those of (4.4). We find

$$\delta(\delta X_2) = [\partial H_2, \lambda^{(2)}] + \partial(\nabla\lambda_1^{(2)}). \quad (4.11)$$

Thus, we introduce a field, $\lambda_2^{(1)} = \frac{1}{2}dx^\mu dx^\nu \lambda_{\mu\nu}^{(1)}$, as in the Abelian case [see (3.8)], but here with the gauge transformation rule

$$\delta\lambda_2^{(1)} = \nabla\lambda_1^{(2)}. \quad (4.12)$$

Then we modify (4.8) to

$$\delta X_2 = \nabla\lambda_1^{(1)} - \frac{1}{2}[Y, \delta Y] - \partial\lambda_2^{(1)}. \quad (4.13)$$

Under the $\lambda^{(1)}$, $\lambda_1^{(1)}$, $\lambda_2^{(1)}$ gauge transformations, we find

$$\delta(\delta X_2) = [\partial H_2, \lambda^{(2)}], \quad (4.14)$$

and so (4.13) transforms covariantly. Note that the term involving $[Y, \delta Y]$ in (4.13) cannot be cancelled by adding some function of g and Y to X_2 . Now we consider (4.12) and find that it does not transform covariantly under the third-order gauge transformations involving $\lambda^{(3)}$ and $\lambda_1^{(3)}$. This leads to the introduction of a new field, $\lambda_2^{(2)}$, etc. We are led to introduce an infinite set of gauge parameter superfields, $\lambda_i^{(i)}$ ($i = 1, 2, \dots$), as in the Abelian case. The non-Abelian generalization of (3.10) is

$$\delta\lambda_2^{(i)} = \nabla\lambda_1^{(i+1)} - \partial\lambda_2^{(i+1)} \quad (4.15)$$

($i = 1, 2, \dots$). It can be checked that these variations are covariant, in that under the $\lambda^{(i+2)}$, $\lambda_1^{(i+2)}$, $\lambda_2^{(i+2)}$ gauge transformations

$$\delta(\delta\lambda_2^{(i)}) = [\partial H_2, \lambda^{(i+2)}]. \quad (4.16)$$

The next step is to find a field strength for the field X_2 , generalizing the Abelian expression (3.13). Some guesswork leads at first to the expression

$$G_3 = dX_2 + \frac{1}{2}[dY, Y] + \frac{1}{3}[\partial Y, Y], \quad (4.17)$$

which transforms under (4.3) and (4.13) as

$$\delta G_3 = [\partial H_2, \lambda_1^{(1)}] + [\delta H_2, Y] + \partial\left(-\nabla\lambda_2^{(1)} + [\delta X_2, Y] + \frac{1}{6}[[Y, \delta Y], Y]\right), \quad (4.18)$$

where δY , δH_2 and δX_2 are as given in (4.3), (4.5) and (4.13), respectively. Consideration of (4.18) leads us to introduce a new field, $X_3 = \frac{1}{6}dx^\mu dx^\nu dx^\rho X_{\mu\nu\rho}$, which transforms under the $\lambda^{(1)}$, $\lambda_1^{(1)}$, $\lambda_2^{(1)}$ gauge transformations as

$$\delta X_3 = \nabla\lambda_2^{(1)} - [\delta X_2, Y] - \frac{1}{6}[[Y, \delta Y], Y]. \quad (4.19)$$

Then we may define

$$H_3 = G_3 + \partial X_3 \quad (4.20)$$

as the X_2 field strength. This transforms as

$$\delta H_3 = [\partial H_2, \lambda_1^{(1)}] + [\delta H_2, Y]. \quad (4.21)$$

The term $[\delta H_2, Y]$ may seem like a noncovariant term, but we can see from (4.13) that the gauge field X_2 has a noncanonical term in its variation δX_2 —the non-Abelian term $\nabla\lambda_1^{(1)}$ and the Abelian term $-\partial\lambda_2^{(1)}$ in δX_2 are analogs of the terms in the variation of the gauge field Y [see (4.3)], but there is an addition a "noncanonical" term involving $[Y, \delta Y]$.

The variation (4.19) does not transform covariantly under the $\lambda^{(2)}$, $\lambda_1^{(2)}$, $\lambda_2^{(2)}$ gauge transformations. We find

$$\delta(\delta X_3) = [\partial H_2, \lambda_1^{(2)}] + \partial(\nabla\lambda_2^{(2)}). \quad (4.22)$$

The last term (4.22) is the noncovariant one, and hence we introduce a new gauge parameter field, $\lambda_3^{(1)}$, which transforms as

$$\delta\lambda_3^{(1)} = \nabla\lambda_2^{(2)}, \quad (4.23)$$

and replace (4.19) by

$$\delta X_3 = \nabla\lambda_2^{(1)} - [\delta X_2, Y] - \frac{1}{6}[[Y, \delta Y], Y] - \partial\lambda_3^{(1)}. \quad (4.24)$$

However, (4.23) does not transform covariantly under the $\lambda^{(3)}$, $\lambda_1^{(3)}$ gauge transformations, etc. Thus we are led to introduce a set of gauge parameter fields $\lambda_i^{(i)}$ ($i = 1, 2, \dots$), which transform as

$$\delta\lambda_3^{(i)} = \nabla\lambda_2^{(i+1)} - \partial\lambda_3^{(i+1)}. \quad (4.25)$$

These transformations are covariant—under the $\lambda^{(i+2)}$, $\lambda^{(i+2)}$, $\lambda^{(i+2)}$ gauge transformations, we have

$$\delta(\delta\lambda_3^{(i)}) = [\partial H_2, \lambda_1^{(i+2)}]. \quad (4.26)$$

Now we seek a field strength for X_3 , etc. The expressions for these field strengths rapidly become complicated. However, the following general structure, which is the non-Abelian generalization of (3.14) and (3.16), appears to emerge.

We have gauge fields $g, X_1 := Y, X_2, \dots, X_n, \dots$, and gauge parameter fields $\lambda_0^{(i)} := \lambda^{(i)}, \lambda_1^{(i)}, \dots, \lambda_n^{(i)}, \dots$ ($i = 1, 2, \dots$) (recall that the subscript on a differential form indicates its degree). The field strength for g is $H_1 = g^{-1}dg + g^{-1}(\partial X_1)g$ [this is the \mathcal{A} of (4.1)]. To find the field strengths for X_n ($n = 1, 2, \dots$), we may proceed inductively. We assume that the field strengths H_2, H_3, \dots, H_n are known, corresponding to the gauge fields X_1, X_2, \dots, X_{n-1} , respectively. The gauge fields X_m ($m = 1, 2, \dots, n$) are assumed to transform under the $\lambda_p^{(i)}$ ($p = 1, 2, \dots, m-1$), $\lambda_m^{(i)}$ gauge transformations as

$$\delta X_m = \nabla \lambda_{m-1}^{(i)} + f_m(\delta X_p, X_p) + (-1)^{m-1} \partial \lambda_m^{(i)}, \quad (4.27)$$

where $f_m(\delta X_p, X_p)$ is some function of X_p and its $\lambda_p^{(i)}$ variations δX_p , for $p = 1, 2, \dots, m-1$. The field strength H_m transforms as

$$\delta H_m = [\partial H_2, \lambda_{m-2}^{(i)}] + \dots, \quad (4.28)$$

where "... " are additional terms deduced from the known form of H_m , using (4.27). To construct the field strength H_{n+1} for X_n , we add suitable functions of X_m to the expression dX_n , to obtain an expression (G_{n+1}) satisfying

$$\delta G_{n+1} = [\partial H_2, \lambda_n^{(i)}] + \dots + (-1)^{n+1} \partial(\nabla \lambda_{n+1}^{(i)} + f_{n+1}(\delta X_m, X_m)), \quad (4.29)$$

where "... " indicates additional noncanonical terms which may arise in this process. [For $n = 1$, H_2 is replaced by G_2 on the right-hand side of (4.29).] Now we propose that X_{n+1} transforms as

$$\delta X_{n+1} = \nabla \lambda_n^{(i)} + f_{n+1}(\delta X_m, X_m) + (-1)^n \partial \lambda_{n+1}^{(i)}. \quad (4.30)$$

Then we define the field strength H_{n+1} by

$$H_{n+1} = G_{n+1} + (-1)^n \partial X_{n+1}. \quad (4.31)$$

This transforms as

$$\delta H_{n+1} = [\partial H_2, \lambda_n^{(i)}] + \dots, \quad (4.32)$$

where "... " are the noncanonical terms of (4.29). The gauge parameter fields $\lambda_n^{(i)}$ ($i = 0, 1, \dots$) are posited to transform as

$$\delta \lambda_n^{(i)} = \nabla \lambda_{n-1}^{(i+1)} + (-1)^{n+1} \lambda_n^{(i+1)}. \quad (4.33)$$

Under the $\lambda_q^{(2)}$ ($q = 1, 2, \dots, n$) gauge transformations, (4.30) transforms covariantly—

$$\delta(\delta X_{n+2}) = [\partial H_2, \lambda_n^{(2)}]. \quad (4.34)$$

Similarly, under the $\lambda_q^{(i+2)}$ ($q = 1, 2, \dots, n$) gauge transformations, (4.33) transforms covariantly—

$$\delta(\delta \lambda_{n+2}^{(i)}) = [\partial H_2, \lambda_n^{(i+2)}]. \quad (4.35)$$

The initial step ($n = 1$) of the above induction procedure takes $X_1 (= Y)$ as the gauge field, transforming as in (4.3). The function f_1 is zero. The field G_2 is given in (4.6), and the function f_2 is given by $f_2(\delta X_1, X_1) = -\frac{1}{2}[X_1, \delta X_1]$. The variation δX_2 is given in (4.13), and H_2 in (4.9). The next inductive step ($n = 2$) is detailed between (4.17) and (4.26). Further inductive steps yield more complicated expressions. The "noncanonical" term $[Y, \delta Y]$ in the variation of X_2 [see (4.13)] causes the field strength of X_2 , i.e. H_3 , to transform with noncanonical terms, which is in turn connected with noncanonical terms in the variation of X_3 , etc. This prevents us from writing an explicit closed form expression for the field strength H_{n+1} and gauge transformation rules δX_n , for arbitrary n .

From the field strengths H_{n+1} , we may derive the BRST for BRST rules corresponding to the gauge transformations (4.30) and (4.33). This follows the same procedure as that used in the Abelian case in the previous section—one defines field strengths $\tilde{H}_{n+1}(x, \theta, \theta_2)$, by replacing untilded with tilded quantities in the definitions of H_{n+1} . One then identifies the ghost fields by the Lie-algebra-valued versions of (3.45), and imposes the constraints (3.46), using the non-Abelian versions of \tilde{H}_{n+1} . With the identification $s_2 = \partial/\partial \theta_2$, this gives the BRST for BRST rules. Let us consider some examples:

The gauge field g has the field strength $\mathcal{A}_1(x, \theta)$, defined in (4.1). We define $\tilde{\mathcal{A}}_1(x, \theta, \theta_2)$ by

$$\tilde{\mathcal{A}}_1(x, \theta, \theta_2) = \tilde{g}^{-1} \tilde{d}\tilde{g} + \tilde{g}^{-1} (\partial \tilde{Y}_1) \tilde{g}, \quad (4.36)$$

where \tilde{g} and \tilde{Y}_1 are functions of (x, θ, θ_2) , with $\tilde{Y}_1 = dx^\mu \tilde{Y}_\mu + d\theta_2 \tilde{Y}_{\theta_2}$. Defining

$$\begin{aligned} g(x, \theta) &= \tilde{g}(x, \theta, 0), & \mathcal{A}_\mu(x, \theta) &= \tilde{\mathcal{A}}_\mu(x, \theta, 0), \\ Y_\mu(x, \theta) &= \tilde{Y}_\mu(x, \theta, 0), & c^{(1)}(x, \theta) &= \tilde{Y}_{\theta_2}(x, \theta, 0). \end{aligned} \quad (4.37)$$

we see that the field strength constraint

$$\bar{\mathcal{A}}_{\theta_2}(x, \theta, \theta_2) = 0 \quad (4.38)$$

evaluated at $\theta_2 = 0$ gives the BRST for BRST transformation

$$s_2 g = (\partial c^{(1)})g. \quad (4.39)$$

The field strength associated with the gauge field $Y_1(x, \theta)$ is given by H_2 in (4.9). We define $H_2(x, \theta, \theta_2)$ by

$$\bar{H}_2 = \bar{\partial} \bar{Y}_1 + \frac{1}{2} [\partial \bar{Y}_1, \bar{Y}_1] - \partial \bar{X}_2. \quad (4.40)$$

Using the obvious Lie-algebra-valued analogs of the definitions (3.32), we find that the constraints

$$\bar{H}_{\theta_2, \mu} = \bar{H}_{\theta_2, \dot{\theta}_2} = 0 \quad (4.41)$$

evaluated at $\theta_2 = 0$ give the BRST for BRST transformations

$$s_2 Y_\mu = \partial_\mu c^{(1)} + \frac{1}{2} [\partial Y_\mu, c^{(1)}] + \frac{1}{2} [\partial c^{(1)}, Y_\mu] - \partial c_\mu^{(1)}, \quad (4.42)$$

$$s_2 c^{(1)} = -\partial c^{(2)} + \frac{1}{2} [\partial c^{(1)}, c^{(1)}].$$

[A field redefinition $c_\mu^{(1)} \rightarrow c_\mu^{(1)} + \frac{1}{2} [Y_\mu, c^{(1)}]$ simplifies the first equation of (4.42), giving $s_2 Y_\mu = \nabla_\mu c^{(1)} + \partial c_\mu^{(1)}$.]

The field strength associated with the gauge field $X_2(x, \theta)$ is $H_3(x, \theta)$, given in (4.20). We define $\bar{H}_3(x, \theta, \theta_2)$ by

$$\bar{H}_3 = \bar{\partial} \bar{X}_2 + \frac{1}{2} [\bar{\partial} \bar{Y}, \bar{Y}] + \frac{1}{3} [\{\partial \bar{Y}, \bar{Y}\}, \bar{Y}] + \partial \bar{X}_3. \quad (4.43)$$

The (x, θ) component fields are defined by Lie-algebra-valued versions of the Abelian definitions (3.38). The Lie-algebra-valued versions of the constraints (3.39) evaluated at $\theta_2 = 0$ then give the BRST for BRST transformation rules

$$s_2 c_\mu^{(1)} = \partial_\mu c^{(2)} + \partial c_\mu^{(2)} - \frac{1}{2} [s_2 c^{(1)}, Y_\mu] - \frac{1}{2} [s_2 Y_\mu, c^{(1)}]$$

$$+ \frac{1}{2} [\partial_\mu c^{(1)}, c^{(1)}] + \frac{1}{3} [\{\partial c^{(1)}, c^{(1)}\}, Y_\mu]$$

$$+ \frac{1}{3} [\{\partial c^{(1)}, Y_\mu\}, c^{(1)}] + \frac{1}{3} [\{\partial Y_\mu, c^{(1)}\}, c^{(1)}], \quad (4.44)$$

$$s_2 c^{(2)} = \partial c^{(3)} - \frac{1}{2} [s_2 c^{(1)}, c^{(1)}] + \frac{1}{3} [\{\partial c^{(1)}, c^{(1)}\}, c^{(1)}], \quad (4.45)$$

plus a complicated equation for $s_2 X_{\mu\nu}$, which we omit for brevity. Using (4.39), (4.42), (4.44) and (4.45), one can check that $(s_2)^2$ applied to α , Y_μ or $c^{(1)}$ gives zero. One can continue applying the above procedure to the field strengths (4.31), imposing as constraints the Lie-algebra-valued versions of (3.46), for $n = 4, 5, \dots$. Using the Lie-algebra-valued versions of the ghost identifications of the Abelian case, (3.45), one may thus derive the full set of BRST for BRST rules. [Anighost and NL superfields are also introduced, transforming as in the Lie-algebra-valued version of (3.48).]

5. Unconstrained BRST for BRST and (BRST)ⁿ

Now we return to the question of the symmetries of the quantum action. The quantum Yang-Mills action, with the Yang-Mills gauge invariance fixed, is given in (2.15). This action is a functional of the prepotentials g and Y_μ (and the antighost \bar{C}^n), and has explicit BRST invariance. It also has the prepotential gauge invariances (2.8) and (2.10). The quantum action, after fixing of these gauge invariances, has a BRST-type symmetry corresponding to the prepotential gauge transformations. This we call "BRST for BRST." This BRST for BRST invariance can be made explicit in the gauge-fixing action which arises from fixing the *prepotential* gauge invariances (this is analogous to the way in which the gauge-fixing part of the BRST-invariant action can be written in an explicitly BRST-invariant form without using prepotentials). However, the gauge-invariant Yang-Mills action and the gauge-fixing action for the Yang-Mills gauge invariance are not in an explicitly BRST-for-BRST-invariant form—again, this is analogous to the BRST case, where we needed to write the Yang-Mills gauge-invariant action in terms of prepotentials [as in (2.13)] for the BRST symmetry to be manifest. Given that our philosophy is to make BRST-type symmetries explicit, we are thus led to apply the "unconstrained BRST" approach to the BRST for BRST symmetry—i.e. we seek to solve the BRST for BRST field strength constraints [the non-Abelian versions of (3.46)] in terms of "prepotentials." Then we will express the quantum action in terms of the prepotentials, and the BRST for BRST invariance will be explicit. It turns out that the resulting prepotential action has additional gauge invariances which arise in the course of solving the BRST for BRST field strength constraints. This leads to a (BRST)³ symmetry, which we can then seek to make explicit by introducing and solving field strength constraints. These solutions will turn out to have gauge invariances, and so on *ad infinitum*. In this section we will show explicitly how this procedure works for the action in the unconstrained BRST for BRST approach. We will then give the form of the gauge-invariant action with explicit (BRST)ⁿ symmetry, with $i = 1, 2, \dots, n$, for any positive integer n .

The simplest three constraints of the non-Abelian versions of (3.46) may be written, using the explicit forms (4.36), (4.40) and (4.43), as

$$H_{\theta_2} = 0 = g^{-1} \partial_{\theta_2} g - g^{-1} \partial Y_{\theta_2} g, \quad (5.1)$$

$$H_{\theta_2 \theta_2} = 0 = -2\partial_{\theta_2} Y_{\theta_2} + \{\partial Y_{\theta_2}, Y_{\theta_2}\} - \partial X_{\theta_2 \theta_2}, \quad (5.2)$$

$$H_{\theta_2 \mu} = 0 = \partial_{\theta_2} Y_{\mu} - \partial_{\mu} Y_{\theta_2} - \frac{1}{2} \{\partial Y_{\theta_2}, Y_{\mu}\} - \frac{1}{2} \{\partial Y_{\mu}, Y_{\theta_2}\} + \partial X_{\theta_2 \mu}. \quad (5.3)$$

We will now solve these constraints by expressing g , Y_{θ_2} , Y_{μ} , $X_{\theta_2 \theta_2}$ and $X_{\theta_2 \mu}$ in terms of pre-potentials. Equation (5.1) has the solution

$$g = hk, \quad h = e^{\partial \gamma}, \quad k = e^{\partial_{\theta_2} \alpha^{(2)}}, \quad (5.4)$$

$$Y_{\theta_2} = \partial Y_{\theta_2}^{(2)} - \int_0^1 dt e^{t\partial} \gamma(\partial_{\theta_2} \gamma) e^{-t\partial} \gamma, \quad (5.5)$$

where γ , $\alpha^{(2)}$ and $Y_{\theta_2}^{(2)}$ are the prepotential superfields, all anticommuting. (The superfields have no dependence upon the variable t .) In order to show that (5.4) and (5.5) give the solution to (5.1), we can use the identity

$$\partial_{\mu} e^x = \int_0^1 dt e^{tx} (\partial_{\mu} x) e^{(1-t)x}, \quad (5.6)$$

where u is some given variable, $\partial_{\mu} = \partial/\partial u$, and x is some matrix-valued (e.g. Lie-algebra-valued) function of u . Using (5.6), we can easily show that

$$\partial Y_{\theta_2} = (\partial_{\theta_2} h) h^{-1} = (\partial_{\theta_2} g) g^{-1}, \quad (5.7)$$

and thus (5.4) and (5.5) solve (5.1). Equation (5.7) expresses the fact that ∂Y_{θ_2} is pure-gauge. Hence, from (5.2) we see that $\partial H_{\theta_2 \theta_2} = 2\partial_{\theta_2} \partial Y_{\theta_2} - [\partial Y_{\theta_2}, \partial Y_{\theta_2}]$ is zero. Whence the expression $-2\partial_{\theta_2} Y_{\theta_2} + \{\partial Y_{\theta_2}, Y_{\theta_2}\}$ is equal to the derivative ∂ applied to some other expression. This expression, which we will not need in an explicit form, may then be taken to be the solution for $X_{\theta_2 \theta_2}$ in terms of the pre-potentials. Thus we have solved (5.2).

To solve (5.3), we first define an anticommuting superfield \hat{Y}_{μ} by

$$\hat{Y}_{\mu} = h(\partial_{\theta_2} Y_{\mu}^{(2)}) h^{-1} - \int_0^1 dt e^{t\partial} \gamma(\partial_{\mu} \gamma) e^{-t\partial} \gamma, \quad (5.8)$$

with $Y_{\mu}^{(2)}$ a commuting pre-potential superfield. Then, by the use of $\partial h = 0$, and (5.6), it follows that

$$\partial \hat{Y}_{\mu} = -\partial_{\mu} h h^{-1} + h(\partial_{\theta_2} Y_{\mu}^{(2)}) h^{-1}. \quad (5.9)$$

Now, we let

$$Y_{\mu} = \partial \epsilon_{\mu} + \hat{Y}_{\mu}, \quad (5.10)$$

with ϵ_{μ} a commuting pre-potential superfield. Then (5.4), (5.5) and (5.10), together with a suitable definition of $X_{\theta_2 \mu}$, solve (5.3). To see this, we note that applying the derivative ∂ to the right-hand side of (5.3) gives zero, using (5.4), (5.5), (5.9) and (5.10). Thus the terms not involving $X_{\theta_2 \mu}$ on the right-hand side of (5.3) equal the derivative ∂ applied to some expression (which we will not need in an explicit form). Taking $X_{\theta_2 \mu}$ to be minus this expression then solves the constraint (5.3).

Now we may use the pre-potential $Y_{\mu}^{(2)}$ to write the gauge-invariant Yang-Mills action in an explicitly BRST- and BRST-for-BRST-invariant form. To do this, we first note that (5.9) relates $\partial \hat{Y}_{\mu}$ and $\partial_{\theta_2} Y_{\mu}^{(2)}$ by a gauge transformation involving h . This leads to the identity

$$\begin{aligned} \partial G_{\mu\nu} &= \partial_{\mu} \partial Y_{\nu} - \partial_{\nu} \partial Y_{\mu} + [\partial Y_{\mu}, \partial Y_{\nu}] \\ &= \partial_{\mu} \partial \hat{Y}_{\nu} - \partial_{\nu} \partial \hat{Y}_{\mu} + [\partial \hat{Y}_{\mu}, \partial \hat{Y}_{\nu}] \\ &= h(\partial_{\mu} \partial_{\theta_2} Y_{\nu}^{(2)}) - \partial_{\nu} \partial_{\theta_2} Y_{\mu}^{(2)} + [\partial_{\theta_2} Y_{\mu}^{(2)}, \partial_{\theta_2} Y_{\nu}^{(2)}] h^{-1} \\ &= h(\partial_{\theta_2} G_{\mu\nu}^{(2)}) h^{-1}, \end{aligned} \quad (5.11)$$

where we define

$$G_{\mu\nu}^{(2)} = \partial_{\mu} Y_{\nu}^{(2)} - \frac{1}{2} [\partial_{\theta_2} Y_{\mu}^{(2)}, Y_{\nu}^{(2)}] - (\mu \leftrightarrow \nu). \quad (5.12)$$

Then, from (5.11),

$$\text{Tr}(\partial G_{\mu\nu} \partial G^{\mu\nu}) = \text{Tr}(\partial_{\theta_2} G_{\mu\nu}^{(2)} \partial_{\theta_2} G^{\mu\nu(2)}), \quad (5.13)$$

and so, considering (2.13), we are led to the following explicitly BRST- and BRST-for-BRST-invariant form of the gauge-invariant Yang-Mills action:

$$S^{\text{inv}} = -\frac{1}{4} \text{Tr} \int d^d x d\theta d_2 G_{\mu\nu}^{(2)} \partial_{\theta_2} G^{\mu\nu(2)}. \quad (5.14)$$

The gauge-fixing actions which fix the Yang-Mills gauge invariance and the prepotential gauge invariances can also be written in an explicitly BRST-for-BRST-invariant form. To see this, we first note that, from (2.6), (5.4), (5.9) and (5.10),

$$\mathcal{A}_\mu = k^{-1} \partial_\mu k + k^{-1} (\partial \theta_2 Y_\mu^{(2)}) k. \quad (5.15)$$

Since $\partial_\theta k = 0$, we deduce that $\partial_\theta \mathcal{A}_\mu = 0$. We introduce the antighost superfield \bar{C}^α as before (see Sec. 2), taking it to be independent of the θ_2 coordinate, i.e. $\partial_{\theta_2} \bar{C}^\alpha = 0$. Thus we see that $\partial_\theta \Psi = 0$, where Ψ is the gauge fermion given in (2.16). Therefore $\Psi = \partial_\theta \Psi^{(1)} = \int d\theta_2 \Psi^{(1)}$, for some $\Psi^{(1)}(x, \theta, \theta_2)$, and using this and (5.14) in (2.15) we see that the quantum Yang-Mills action, with the Yang-Mills gauge invariance fixed, can be written in an explicitly BRST-for-BRST-invariant form. Now, we consider the action which fixes the prepotential gauge invariances, given by the term $s_2 \Psi^{(2)}$, in (2.17), an explicit example of $\Psi^{(2)}$ being given in (2.19). The ghosts are given in terms of the prepotentials by evaluating at $\theta_2 = 0$ the (x, θ, θ_2) superfield solutions to the BRST for BRST constraints. These we have just discussed—e.g. $c^{(1)}$ is given by the expression Y_{θ_2} in (5.5), $c^{(2)}$ and $c_\mu^{(1)}$ by the expressions for $\frac{1}{2} X_{\theta_2 \theta_2}$ and $X_{\theta_2 \mu}$, respectively, etc. The BRST for BRST antighosts and NL fields fit into (x, θ, θ_2) superfields as $\bar{C}_\mu^{(1)}(x, \theta, \theta_2) = \bar{C}_\mu^{(1)}(x, \theta) + \theta_2 \pi_\mu^{(1)}(x, \theta)$ [see (3.48)]. Using these superfield expressions and $s_2 = \partial_\theta = \int d\theta_2$, we may thus write the term $s_2 \Psi^{(2)}$ in an explicitly BRST-for-BRST-invariant form.

Therefore, we have arrived at an explicitly BRST-for-BRST-invariant form of the quantum Yang-Mills action, given by (2.17), written in (x, θ, θ_2) superspace as just discussed. This action is a functional of the pre-prepotentials (and the antighost superfields). However, this cannot be taken to be the final form of the quantum Yang-Mills action, as it has pre-prepotential gauge invariances—these are any variations of the pre-prepotentials which preserve the solutions (5.4), (5.5), (5.10), etc. [these clearly are pre-prepotential gauge invariances—e.g. in (5.10), $\delta \epsilon_\mu = \partial \omega_\mu$, with ω_μ an anticommuting superfield]. These pre-prepotential gauge invariances are the BRST for BRST analogs of the prepotential gauge invariances (2.8). Here we just wish to point out that these pre-prepotential gauge invariances will be symmetries of the BRST for BRST gauge-fixed action. The latter can be written as in (2.17). From this form we can see that it is a functional only of the (x, θ, θ_2) superspace prepotentials (and their θ_2 derivatives), evaluated at $\theta_2 = 0$. As these prepotentials are invariant under the pre-prepotential gauge invariances, so is this action. (The antighost superfields are taken to be invariant under the pre-prepotential gauge transformations.)

The pre-prepotential gauge invariances have corresponding $(\text{BRST})^3$ transformations, and we may apply the unconstrained BRST approach to these. As discussed above, we are led eventually to $(\text{BRST})^n$ for any positive integer n . This follows the same procedure as that used in the case of BRST for BRST. With regard to the gauge-invariant Yang-Mills action, we find that consideration of the next few stages, together with the results already given, points to the following formulation for $(\text{BRST})^n$:

The fundamental gauge field is a Lie-algebra-valued vector field $Y_\mu^{(n)}$, which is anticommuting if n is odd and commuting if n is even, for a positive integer. The gauge-invariant Yang-Mills action is given by

$$\begin{aligned} S^{\text{inv}} &= -\frac{1}{4} \text{Tr} \int d^d x F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{4} \text{Tr} \int d^d x d^n \theta G_{\mu\nu}^{(n)} \partial^{(n)} G^{\mu\nu(n)}, \end{aligned} \quad (5.16)$$

where we define $d^n \theta = \prod_{i=1}^n d\theta_i$, $\partial^{(n)} = \prod_{i=1}^n \partial/\partial\theta_i$, and

$$G_{\mu\nu}^{(n)} = \partial_\mu Y_\nu^{(n)} - \frac{1}{2} [\partial^{(n)} Y_\mu^{(n)}, Y_\nu^{(n)}] - (\mu \leftrightarrow \nu). \quad (5.17)$$

The gauge field $Y_\mu^{(n)}$ is a function of x and θ_i ($i = 1, 2, \dots, n$). The action (5.16) has explicit $(\text{BRST})^i$ invariance, for $i = 1, 2, \dots, n$ —since the generators of these symmetries are translations in the θ_i directions.

6. Final Remarks

To summarize, we have studied the formulation of quantum Yang-Mills theory given in Refs. 8 and 9, in which the BRST symmetry of the quantum action is made manifest by writing the theory in terms of the prepotential gauge fields which solve the BRST curvature constraints. We have investigated the infinitely reducible gauge symmetry which this prepotential quantum Yang-Mills action possesses. The BRST transformations corresponding to this infinitely reducible gauge invariance were discussed. These “BRST for BRST” transformations are invariances of the prepotential quantum Yang-Mills action which arises after fixing of the infinitely reducible gauge symmetries. In line with the philosophy of making BRST-type invariances explicit, we were thus led to apply the unconstrained BRST approach of Refs. 8 and 9 to the BRST for BRST invariance. This led to a form of the quantum Yang-Mills action with explicit BRST for BRST symmetry, obtained by using the pre-prepotentials which solved the BRST for BRST curvature constraints. This action in turn has prepotential gauge invariances, with corresponding $(\text{BRST})^3$ transformations, leading to a form of the Yang-Mills action with explicit $(\text{BRST})^3$ invariance. This procedure may be continued indefinitely, and we gave the form of the gauge-invariant Yang-Mills action with explicit $(\text{BRST})^i$ ($i = 1, 2, \dots, n$) invariance, for any positive integer n .

A number of comments may be made. We could clearly apply this approach to the unconstrained BRST formulation of other gauge theories, given in Ref. 9. We expect a similar structure. In Sec. 4 we gave an inductive procedure for finding the non-Abelian field strengths H_{n+1} for the gauge fields X_n which were introduced. Explicit closed form expressions for the H_{n+1} would be preferable. However, these expressions may be quite complicated, or given only in an implicit form—this is suggested first by explicit calculation of some further cases, but also by trying to deduce the non-Abelian versions of (3.4), beginning with (4.39), (4.42), (4.44) and (4.45), and requiring $(s_2)^2 = 0$ on all fields. The resulting expressions for the BRST for BRST

rules become complicated. As these transformation rules can be expressed as the vanishing of certain components of the field strengths, this too indicates that the explicit forms of the field strengths are similarly complicated. Possibly an augmented set of auxiliary fields will simplify the situation. One might also seek a unified understanding of the system of auxiliary fields which we introduced.

It was pointed out in Sec. 2 that the obvious gauge-fixing conditions for the prepotential gauge invariances resulted in the breaking of BRST symmetry. The resulting quantum action nevertheless had a BRST for BRST symmetry. This was then made explicit using pre-potentials. Gauge-fixing the invariances of the latter similarly seems to involve breaking the explicit BRST for BRST symmetry, the resulting quantum action having a $(\text{BRST})^3$ symmetry, however. This procedure continues indefinitely, as we have seen. This is an obstacle to formulating quantum Yang-Mills theory in a way in which BRST symmetries are manifest, in that applying the unconstrained BRST approach to the action with $(\text{BRST})^n$ symmetry indeed makes this symmetry explicit, however gauge invariances arise in this process, and gauge-fixing these seems to involve breaking the explicit $(\text{BRST})^n$ symmetry. This problem deserves further investigation. Certainly, as we have seen, there is a surprisingly rich structure to be found when one pursues the unconstrained BRST approach.

Acknowledgment

It is a pleasure to thank Chris Hull and José Luis Vázquez-Bello for helpful conversations.

References

1. C. Becchi, A. Rouet and R. Stora, *Phys. Lett.* **52B** (1974) 344, *Commun. Math. Phys.* **42** (1975) 127; I. V. Tyutin, "Gauge Invariance in Field Theory and in Statistical Mechanics in the Operator Formalism," Lebedev preprint FIAN 39 (1975), unpublished.
2. L. Baulieu and J. Thierry-Mieg, *Nucl. Phys.* **B197** (1982) 477.
3. L. Baulieu, *Phys. Rep.* **129** (1985) 1.
4. E. S. Fradkin and G. A. Vilkovisky, *Phys. Lett.* **B55** (1975) 224; I. A. Batalin and G. A. Vilkovisky, *Phys. Lett.* **B69** (1977) 309.
5. M. Henneaux, *Phys. Rep.* **126** (1985) 1.
6. J. Thierry-Mieg, *J. Math. Phys.* **21** (1980) 2834, *Nuovo Cimento* **56A** (1980) 396; L. Bonora and M. Tonin, *Phys. Lett.* **98B** (1981) 48; J. Thierry-Mieg and Y. Ne'eman, *Ann. Phys.* **123** (1980) 247; M. Quiros, F. J. de Urries, J. Hoyos, M. J. Mazon and E. Rodríguez, *J. Math. Phys.* **22** (1981) 1767; L. Bonora, P. Pasti and M. Tonin, *Nuovo Cimento A64* (1981) 307; R. Delbourgo and P. D. Jarvis, *J. Phys. A. (Math. Gen.)* **15** (1982) 611; A. Hirschfeld and H. Leschke, *Phys. Lett.* **101B** (1981) 48; L. Baulieu, *Nucl. Phys.* **B241** (1984) 557.
7. S. Ferrara, O. Piguet and M. Schweda, *Nucl. Phys.* **B119** (1977) 493; K. Fujikawa, *Prog. Theor. Phys.* **59** (1978) 2045, **63** (1980) 1364; L. Bonora, P. Pasti and M. Tonin, *Ann. Phys.* **144** (1982) 15; P. D. Delbourgo, P. D. Jarvis and G. Thompson, *Phys. Lett.* **109B** (1982) 25, *Phys. Rev.* **D26D** (1982) 775.
8. C. M. Hull, B. Spence and J. L. Vázquez-Bello, *Nucl. Phys.* **B348** (1991) 108.
9. E. Dur and S. James Gates Jr., *Nucl. Phys.* **B343** (1990) 622.

NEW PERTURBATIVE APPROACH TO GENERAL RENORMALIZABLE QUANTUM FIELD THEORIES

V. GUPTA

Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

and

D. V. SHIRKOV and O. V. TARASOV

Joint Institute for Nuclear Research, Dubna, USSR

Received 17 July 1990

We develop further the new approach to perturbation theory for renormalizable quantum field theories (proposed some years ago) which gives renormalization-scheme-independent predictions for observable quantities. We call the resulting Renormalization-Scheme-Independent Perturbation theory RESIPE, for short.

First, we formulate explicitly the relation of RESIPE to the renormalization group formalism for the massless one-coupling case. Then we extend this to the case where particle masses cannot be neglected. Further, we generalize the RESIPE formalism for the theory with two coupling constants. A new scheme-invariant perturbation expansion, without reference to renormalization group techniques, is given which is valid for the general case with masses, several kinematic variables and more than one coupling constant. In conclusion, we argue that the appropriately generalized RESIPE provides us with a picture of perturbative predictions, for renormalizable quantum field theories, that is free from regularization and renormalization scheme ambiguities.

1. Introduction

In the traditional framework of the renormalizable theory of quantum fields, the results of perturbative calculations are usually expressed in terms of renormalized Lagrangian parameters, i.e. coupling constants and masses, within a definite renormalization scheme (RS). Due to this, expansion parameters (coupling constants or running couplings) as well as expansion coefficients are scheme-dependent. In addition, in gauge theories they are gauge-dependent. The problem of scheme dependence turns out to be important quantitatively for the cases where "physical" renormalizable coupling is not very small numerically. Such is the case in quantum chromodynamics (QCD).

A few years ago, in a set of papers,¹⁻³ a new perturbative approach to renormalizable quantum field theories (QFTs) was proposed. This approach yields finite perturbative predictions, which are free from RS ambiguities, for a physical quantity. Further, it can be applied to an object constructed from a Green's function so that it is not explicitly renormalized. We shall call it² RESIPE (see abstract).

The central idea of RESIPE is to use some observable quantity as the perturbation expansion parameter instead of the usual RS-dependent coupling constant, as is normally done. It is because of this key ingredient, namely expanding a physical